Table 1 Estimation error and loss function

	Matrix method		New method	
Case	$\phi_{err}(A_{opt})$, rad	$L(A_{opt})$	$\phi_{err}(A_{est})$, rad	$L(A_{est})$
1	1.23×10^{-6}	1.81	1.23×10^{-6}	1.81
2	1.79×10^{-6}	1.15	1.79×10^{-6}	1.15
3	1.25×10^{-2}	1.86	1.24×10^{-2}	1.86
4	1.81×10^{-2}	1.18	1.80×10^{-2}	1.18
5	1.21×10^{-2}	0.07	1.21×10^{-2}	0.07
6	3.10×10^{-5}	2.19	3.10×10^{-5}	2.19
7	3.94×10^{-5}	1.70	3.94×10^{-5}	1.70
8	0.235	2.26	0.192	2.42
9	0.105	1.78	0.205	1.86
10	2.17×10^{-2}	2.13	2.11×10^{-2}	2.13
11	4.22×10^{-2}	0.13	4.21×10^{-2}	0.13
12	2.74×10^{-2}	2.34	2.84×10^{-2}	2.34

measurements, and a smaller error in the other case. The estimation errors of the two methods are the same in cases with $\boldsymbol{n}_i = \boldsymbol{0}$, but the new method gives a more accurately orthogonal attitude matrix. It is very nice that no significant loss of accuracy results from omitting the optimization of λ, since this optimization can be time consuming and very sensitive to numerical inaccuracies.[6]

The computational speed of the new method was also compared with FOAM and with QUEST,[4] which are the fastest known methods for minimizing Wahba's loss function. The measured CPU times were computed for sets of 2 to 12 observations similar to those given by Eq. (28). They consist of a part that is independent of the number of observations and a part proportional to the number of observations

$$t_{new} = 0.28 + 0.07n \text{ msec} \tag{31}$$

$$t_{FOAM} = 0.26 + 0.07n \text{ msec for } \sigma_i = 10^{-6}\text{ rad} \tag{32}$$

$$t_{FOAM} = 0.32 + 0.07n \text{ msec for } \sigma_i = 0.01\text{ rad} \tag{33}$$

$$t_{QUEST} = 0.24 + 0.09n \text{ msec for } \sigma_i = 10^{-6}\text{ rad} \tag{34}$$

$$t_{QUEST} = 0.30 + 0.09n \text{ msec for } \sigma_i = 0.01\text{ rad} \tag{35}$$

The n-dependent time is the time required to normalize the input vectors and form the B matrix. The n-independent time is the time required to perform all other computations, including computation of the attitude error angle covariance matrix.[4,6] The time required for the iterative optimization of λ in FOAM or QUEST is seen to depend on the level of measurement errors. The new method, on the other hand, is executed in a fixed amount of time, since a fixed number of mathematical operations is performed. The absolute execution times are not of great significance; only the relative times are interesting. In any case, the time required for any of the three methods appears to be quite modest in comparison with other computations performed in spacecraft attitude determination.

Conclusions

A new quaternion estimation algorithm that finds an approximate minimum of Wahba's loss function has been shown to provide attitude estimates as accurate as the optimal estimates. The new algorithm is also as computationally efficient and robust as existing algorithms in the test cases examined. The attitude matrix and the quaternion are inherently nonsingular, and any potential problems with special cases like 180-deg rotations are avoided by using Shepperd's algorithm to extract the quaternion from the attitude matrix. A significant advantage of the new method for onboard applications is that it involves no iterative processes that require an uncertain amount of execution time and present the possibility of divergence.

References

[1]Markley, F. L., "Parameterization of the Attitude," *Spacecraft Attitude Determination and Control,* edited by J. R. Wertz, D. Reidel, Dordrecht, The Netherlands, 1978, pp. 410–420.

[2]Murrell, J. W., "Precision Attitude Determination for Multimission Spacecraft," AIAA Guidance, Navigation, and Control Conf., AIAA Paper 78-1248, Palo Alto, CA, Aug. 1978.

[3]Lefferts, E. J., Markley, F. L., and Shuster, M. D., "Kalman Filtering for Spacecraft Attitude Estimation," *Journal of Guidance, Control, and Dynamics,* Vol. 5, No. 5, 1982, pp. 417–429.

[4]Shuster, M. D., and Oh, S. D., "Three-Axis Attitude Determination from Vector Observations," *Journal of Guidance and Control,* Vol. 4, No. 1, 1981, pp. 70–77.

[5]Wahba, G., "A Least Squares Estimate of Spacecraft Attitude," *SIAM Review,* Vol. 7, No. 3, 1965, p. 409.

[6]Markley, F. L., "Attitude Determination Using Vector Observations: A Fast Optimal Matrix Algorithm," *Journal of the Astronautical Sciences,* Vol. 41, No. 2, 1993, pp. 261–280.

[7]Golub, G. H., and Van Loan, C. F., *Matrix Computations,* Johns Hopkins Univ. Press, Baltimore, MD, 1983.

[8]Horn, R. A., and Johnson, C. R., *Matrix Analysis,* Cambridge Univ. Press, Cambridge, England, UK, 1985.

[9]Davenport, P. B., "Attitude Determination and Sensor Alignment via Weighted Least Squares Affine Transformations," AAS/AIAA Astrodynamics Specialists Conference, AAS Paper 71-396, Ft. Lauderdale, FL, Aug. 1971.

[10]Bar-Itzhack, I. Y., and Fegley, K. A., "Orthogonalization Techniques of a Direction Cosine Matrix," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. AES-5, No. 5, 1969, pp. 798–804.

[11]Björck, A., and Bowie, C., "An Iterative Algorithm for Computing the Best Estimate of an Orthogonal Matrix," *SIAM Journal on Numerical Analysis,* Vol. 8, No. 2, 1971, pp. 358–364.

[12]Giardina, C. R., Bronson, R., and Wallen, L., "An Optimal Normalization Scheme," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. AES-11, No. 4, 1975, pp. 443–446.

[13]Higham, N. J., "Computing the Polar Decomposition-with Applications," *SIAM Journal on Scientific and Statistical Computing,* Vol. 7, No. 4, 1986, pp. 1160–1174.

[14]Markley, F. L., "Attitude Determination Using Vector Observations and the Singular Value Decomposition," *Journal of the Astronautical Sciences,* Vol. 36, No. 3, 1988, pp. 245–258.

[15]Shepperd, S. W., "Quaternion from Rotation Matrix," *Journal of Guidance and Control,* Vol. 1, No. 3, 1978, pp. 223, 224.

Rendezvous Guidance with Proportional Navigation

Pin-Jar Yuan* and Shih-Che Hsu†

Chung Shan Institute of Science and Technology, Lungtan, Taiwan 32526, Republic of China

I. Introduction

PROPORTIONAL navigation has been widely used as the guidance scheme in the homing phase of flight for most missile systems.[1-13] Under these guidance schemes, the component of relative velocity in the direction normal to the line-of-sight (LOS) between an interceptor and its target is always driven to zero during intercept course, and the component of relative velocity along the LOS is not required to approach zero for effective intercept of target. However, for the rendezvous problem of two space vehicles, the relative velocity must be driven to zero when the two vehicles meet. For this reason the commanded acceleration of the active vehicle must be applied in both the direction normal to LOS and the direction along LOS simultaneously to reduce the relative velocity to zero as the two vehicles approach each other. Besides rendezvous guidance laws studied before,[14-18] a new rendezvous guidance scheme for use on the terminal phase in an approach

Received April 25, 1992; revision received Feb. 12, 1993; accepted for publication June 3, 1993.

*Senior Scientist, System Development Center. Member AIAA.

†Senior Scientist, System Development Center.

to a space station or satellite is presented in this Note. Under this algorithm, the commanded acceleration is applied in the direction with a bias angle to the normal direction of LOS in general, and its magnitude is programmed by the modified proportional navigation guidance law.[8] Then the solutions of this rendezvous path can be easily obtained as a function of range-to-go or line-of-sight angle under this guidance scheme. The corresponding energy expenditure in exoatmospheric rendezvous, which is related to propellant mass required, is also derived.

II. Analysis

For the case of planar rendezvous guidance, an active spacecraft maneuvers with thrust controlled in both amplitude and direction and tries to rendezvous a passive nonmaneuvering target (for example, a space station or satellite) in exoatmospheric flight. Here a polar coordinate with its center at the target point is defined, and the effect of gravity difference is neglected on the terminal phase of rendezvous. Then the equations of relative motion are

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_\theta \tag{1a}$$

$$\ddot{r} - r\dot{\theta}^2 = a_r \tag{1b}$$

where r is the range-to-go between spacecraft and target, θ is the angle between LOS and inertial reference line, and a_θ, a_r are two components of the total commanded accelerations $\boldsymbol{a}_c$ in both the direction normal to LOS and the direction along LOS, respectively, as depicted in Fig. 1. Now a_θ is programmed with the modified proportional navigation,[8] and a_r is regulated proportional to the centripetal acceleration $r\dot{\theta}^2$ first, which can be also transformed to be the same as that of the modified proportional navigation later, i.e.,

$$a_\theta = \lambda_\theta \dot{r}\dot{\theta} \tag{2a}$$

$$a_r = \lambda_r r\dot{\theta}^2 \tag{2b}$$

where λ_θ, λ_r are constants and dependent to each other under the constraint for rendezvous. From Eq. (1a), the solution of $\dot{\theta}$ can be obtained as a function of r

$$\dot{\theta} = \dot{\theta}_0 (r/r_0)^{\lambda_\theta - 2} \tag{3}$$

where $\dot{\theta}_0$, r_0 are the initial conditions of $\dot{\theta}$ and r, respectively. Substituting the result of Eq. (3) into Eq. (1b) and integrating with respect to r, then the solution of $\dot{r}$ can be derived as

$$\dot{r}^2 = \frac{\lambda_r + 1}{\lambda_\theta - 1} r_0^2 \dot{\theta}_0^2 \left(\frac{r}{r_0}\right)^{2(\lambda_\theta - 1)} + \dot{r}_0^2 - \frac{\lambda_r + 1}{\lambda_\theta - 1} r_0^2 \dot{\theta}_0^2 \tag{4}$$

where $\dot{r}_0$ is the initial rate of range-to-go. For effective rendezvous, the final relative velocity must reach zero when range-to-go goes to zero. Thus the constraint between λ_r and λ_θ is

$$\lambda_r = (\lambda_\theta - 1)A^2 - 1 \tag{5}$$

with $A = \dot{r}_0 / r_0 \dot{\theta}_0$. A positive λ_r is acquired when $(\lambda_\theta - 1)A^2 > 1$ and a negative λ_r is required with $(\lambda_\theta - 1)A^2 < 1$. Under this guidance algorithm, we then have

$$\dot{r} = \dot{r}_0 (r/r_0)^{\lambda_\theta - 1} \tag{6}$$

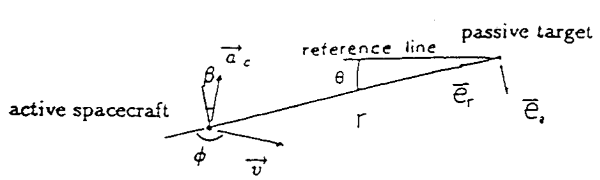

Fig. 1 Planar rendezvous geometry.

We have seen that λ_θ must be chosen to be greater than one and the initial range-to-go rate must be negative. Also, the constraint between λ_θ and λ_r as depicted in Eq. (5) must be satisfied. During the course of the rendezvous, the variation of the relative velocity in both components is similar as a function of r with exponential order $\lambda_\theta - 1$, so the angle $\phi(=\tan^{-1} r\dot{\theta}/\dot{r})$ between LOS and the relative velocity vector maintains unchanged. And the variation of LOS angle θ can be derived as

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{\dot{\theta}}{\dot{r}} \, dr = \frac{1}{A} \ln \frac{r}{r_0} \tag{7a}$$

or

$$r = r_0 e^{A(\theta - \theta_0)} \tag{7b}$$

Thus the relative rendezvous trajectory is a form of spiral function (logarithmic). And the total time of flight until rendezvous can also be derived as

$$T = \int_{r_0}^{0} \frac{dr}{\dot{r}} = \frac{r_0}{(\lambda_\theta - 2)\dot{r}_0} \qquad \text{if } \lambda_\theta < 2 \tag{8a}$$

$$= \infty \qquad \text{if } \lambda_\theta \geq 2 \tag{8b}$$

Therefore, for effective rendezvous in finite time, $1 < \lambda_\theta < 2$ must be satisfied.

Furthermore, the total commanded acceleration is always applied in a fixed direction with a bias angle β relative to the direction normal to LOS during the course of the rendezvous, and its magnitude is proportional to $\dot{r}\dot{\theta}$ (or $r\dot{\theta}^2$), i.e.,

$$a_c = \lambda \dot{r}\dot{\theta} = \sqrt{(\lambda_\theta - 1)^2 (1 + A^2) + \frac{1 + A^2}{A^2}} \, \dot{r}\dot{\theta} \tag{9a}$$

$$\tan \beta = \frac{a_r}{a_\theta} = \frac{(\lambda_\theta - 1)A^2 - 1}{\lambda_\theta A} \tag{9b}$$

It shows that the commanded acceleration approaches zero when $\lambda_\theta > 3/2$ and approaches infinity when $\lambda_\theta < 3/2$ during the course of the rendezvous. Here the bias angle β required for the commanded acceleration vector is simply a function of the selected proportional navigation constant λ_θ (or λ) and the initial condition A, as depicted in Fig. 2. It approaches the opposite direction of relative velocity when λ_θ goes to infinity and approaches the direction normal to relative velocity when λ_θ goes to one. Here λ_θ can also be expressed as a function of the total effective proportional navigation constant λ

$$\lambda_\theta = \sqrt{\frac{\lambda^2 - (1 + 1/A^2)}{1 + A^2}} + 1 \tag{10}$$

where λ must be selected greater than $\sqrt{1 + 1/A^2}$.

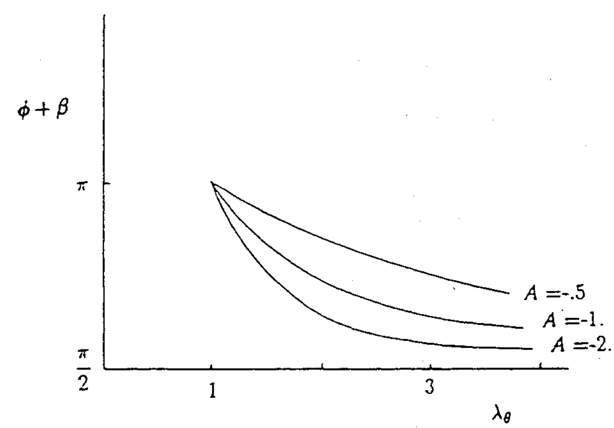

Fig. 2 $\phi + \beta$ vs λ_θ for different initial conditions.

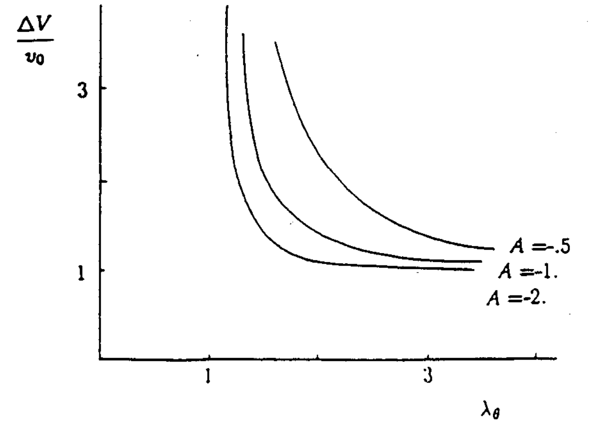

Fig. 3 Energy cost for different initial conditions.

Also, the total energy expenditure required for rendezvous is

$$\Delta V = \int_0^T |a_c| \, \mathrm{d}t = v_0 \sqrt{1 + \frac{1}{\dfrac{A^2}{1+A^2}\lambda^2 - 1}} \tag{11}$$

where $v_0 \; (=\sqrt{\dot{r}_0^2 + r_0^2\dot{\theta}_0^2})$ is the magnitude of initial relative velocity. The ΔV required for exoatmospheric rendezvous is proportional to and always greater than the initial relative velocity. For a smaller value of $|A|$ or λ, a larger energy cost is required, as depicted in Fig. 3.

III. Conclusion

In this rendezvous guidance, the magnitude of the commanded acceleration is programmed similarly to that of modified true proportional navigation, but it is applied with a bias angle to the direction normal to LOS in general. Under this guidance scheme, the closed-form solution of the relative motion described in a polar coordinate can be simply obtained as a function of range-to-go or LOS angle. The relative rendezvous path is a form of spiral function, which means that the direction of the relative velocity is unchanged relative to the LOS, and the total commanded acceleration is always applied in a fixed direction related to the relative velocity during the course of the rendezvous. Also, the energy expenditure required is proportional to and greater than the initial magnitude of relative velocity.

References

[1]Guelman, M., "A Qualitative Study of Proportional Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-7, No. 4, 1971, pp. 337–343.

[2]Guelman, M., "Proportional Navigation with a Maneuvering Target," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-8, No. 3, 1972, pp. 364–371.

[3]Guelman, M., "Missile Acceleration in Proportional Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-9, No. 3, 1973, pp. 462–463.

[4]Becker, K., "Closed-Form Solution of Pure Proportional Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, AES-26, No. 3, 1990, pp. 526–533.

[5]Murtaugh, S. A., and Criel, H. E., "Fundamentals of Proportional Navigation," *IEEE Spectrum*, Vol. 3, No. 6, 1966, pp.75–85.

[6]Guelman, M., "The Closed-Form Solution of True Proportional Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, AES-12, No. 4, 1976, pp. 472–482.

[7]Cochran, J. E., Jr., No, T. S., and Thaxton, D. G., "Analytical Solutions to a Guidance Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 1, 1991, pp. 117–122.

[8]Yuan, P. J., and Chern, J. S., "Solutions of True Proportional Navigation for Maneuvering and Non-maneuvering Targets," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 1, 1992, pp. 268–271.

[9]Yang, C. D., Yeh, F. B., and Chen, J. H., "The Closed-Form Solution of Generalized Proportional Navigation," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 2, 1987, pp. 216–218.

[10]Yang, C. D., Hsiao, F. B., and Yeh, F. B., "Generalized Guidance Law for Homing Missiles," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 25, No. 2, 1989, pp. 197–212.

[11]Yuan, P. J., and Hsu, S. C., "Exact Closed-Form Solution of Generalized Proportional Navigation," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 5, 1993, pp. 963–966.

[12]Yuan, P. J., and Chern, J. S., "Ideal Proportional Navigation," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 5, 1992, pp. 1161–1165.

[13]Yuan, P. J., and Chern, J. S., "Analytic Study of Biased Proportional Navigation," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 1, 1992, pp. 185–190.

[14]Guelman, M., "Guidance for Asteroid Rendezvous," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 5, 1991, pp. 1080–1083.

[15]Jensen, D. L., "Kinematics of Rendezvous Maneuvers," *Journal of Guidance, Control, and Dynamics*, Vol. 7, No. 3, 1984, pp. 307–314.

[16]Niemi, N. J., "Investigation of a Terminal Guidance System for a Satellite Rendezvous," *AIAA Journal*, Vol. 1, No. 2, 1963, pp. 405–411.

[17]Cicolani, L. S., "Trajectory Control in Rendezvous Problems Using Proportional Navigation," NASA TN-D-772, April 1961.

[18]Steffan, K. F., "Satellite Rendezvous Terminal Guidance System," *American Rocket Society Journal*, Vol. 31, No. 11, 1961, pp. 1516–1521.

Elastoplastic Analysis and Cumulative Damage Study of a Lanyard Under Dynamic Conditions

B. S. Nataraju* and B. P. Nagaraj†

ISRO Satellite Centre, Bangalore 560 017, India

Introduction

THE coilable lattice boom (CLB) shown in Fig. 1a is used to deploy a solar sail. Figure 1b shows the sail stowed against the deck of the satellite and held by six launch restraint rods. One end of each launch restraint rod is attached to the spider block and the other end to the hold down assembly. Figure 1c shows the pop-out spring, housed inside the spider block. The CLB is stowed inside a canister with a preloaded tip plate. A lanyard made of thin beryllium-copper strip is wound on a motor-driven pulley with one end attached to the tip plate. The rate of deployment of the CLB is controlled by the lanyard. In the stowed configuration the lanyard is slack. When the tie rod is cut by a pyrocutter, the whole system jumps freely through 0.02 m and then the lanyard becomes taut.

The CLB has a large strain energy stored due to the coiling of elements. The spider block with pop-out spring, the launch restraint rods, and the tip plate possess large strain energy. This energy is absorbed by the lanyard after the system is released.

This Note addresses the formulation of equations of motion for this system to assess the deformation and the load acting on the lanyard. An elastoplastic analysis is necessary as the lanyard is yielding after first release. The system will be stowed back and released for different test configurations before launch. The lanyard undergoes a permanent deformation after each release. Hence, the cumulative damage at the end of each release is calculated. Furthermore, the number of cycles the lanyard would withstand before failure is assessed to ensure that the lanyard is intact during the on-orbit deployment.

Received Aug. 12, 1991; revision received Dec. 1, 1992; accepted for publication Dec. 23, 1992.

*Head, Dynamics and Analysis Section, Spacecraft Mechanisms Division.

†Engineer, Spacecraft Mechanisms Division.